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Eulerian and Lagrangian time microscales in isotropic turbulence

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In isotropic 'box' turbulence without a mean flow, the Lagrangian frequency spectrum extends to frequencies of order $(\epsilon/\nu)^{\frac{1}{2}}$ (ϵ is the rate of dissipation of kinetic energy per unit mass and ν is the kinematic viscosity of the fluid). This leads to an estimate that makes the r.m.s. value of du/dt of order $(\epsilon^3/\nu)^{\frac{1}{4}}$. The Eulerian frequency spectrum, however, extends to higher frequencies than its Lagrangian counterpart; this is caused by spectral broadening associated with large-scale advection of dissipative eddies. As a consequence, the r.m.s. value of $\partial u/\partial t$ at a fixed observation point is (apart from a numerical factor) $R_\lambda^{\frac{1}{2}}$ times as large as the r.m.s. value of du/dt (R_λ is the turbulence Reynolds number based on the Taylor microscale). The results of a theoretical analysis based on these premises agree with data obtained by Comte-Bellot, Shlien and Corrsin. The analysis also suggests that the Eulerian frequency spectrum has a $\omega^{-\frac{5}{3}}$ behaviour in the inertial subrange, and that it is not governed by Kolmogorov similarity.

1. Introduction

The shapes of the Eulerian and Lagrangian frequency spectra in isotropic 'box' turbulence without mean flow have been subjects of some speculation over the years (Inoue 1951; Corrsin 1963; Tennekes & Lumley 1972). Both spectra are assumed to obey Kolmogorov scaling; this leads to forms which are proportional to $\epsilon\omega^{-2}$ in the inertial subrange (ϵ is the rate of dissipation of kinetic energy per unit mass and ω is the angular velocity).

Favre, Gaviglio & Dumas (1951) and Comte-Bellot & Corrsin (1971) measured the Eulerian time correlation function (in a frame of reference moving with the mean flow) of isotropic wind-tunnel turbulence. From their experimental results, Comte-Bellot & Corrsin calculated the Eulerian time microscale. The calculated value, however, was five times smaller than the one derived from the hypothetical similarity between the Eulerian and Lagrangian frequency spectra. This discrepancy suggests that the assumptions involved in the theoretical models should be re-examined. In this paper, the results of an alternative theoretical approach are presented.

The highest frequencies characterizing the dynamics of turbulence occur at the smallest length scales. The Kolmogorov microscale is $(\nu^3/\epsilon)^{\frac{1}{4}}$, the Kolmogorov frequency of dissipative eddies is $(\epsilon/\nu)^{\frac{1}{2}}$ and the kinetic energy of the dissipative eddies is of order $(\nu\epsilon)^{\frac{1}{2}}$ per unit mass. It appears reasonable to postulate that the position of the viscous cut-off in the Lagrangian frequency spectrum is determined by the parameters ν and ϵ.

An Eulerian observer of 'box' turbulence, however, will on occasion encounter appreciable energy at frequencies much larger than $(\epsilon/\nu)^{\frac{1}{2}}$. Random advection of the dissipative structure past the observation point causes spectral broadening, which is not unlike a Doppler effect. The highest frequencies that will be observed must be associated with the advection of dissipative eddies past the observation point by the most energetic eddies. The frequencies involved must be of order q/η, where $\frac{1}{2}q^2$ is the mean kinetic energy per unit mass and η is the Kolmogorov microscale $(\nu^3/\epsilon)^{\frac{1}{4}}$. A simple calculation, based on the assumption that $\epsilon \sim q^3/l$ (where l is an integral scale), shows that q/η is larger than $(\epsilon/\nu)^{\frac{1}{2}}$ by a factor proportional to $R_l^{\frac{1}{4}} = (ql/\nu)^{\frac{1}{4}}$.

In turbulence at high Reynolds numbers, therefore, the dissipative eddies flow past an Eulerian observer in a time much shorter than the time scale which characterizes their own dynamics. This suggests that Taylor's 'frozen-turbulence' approximation should be valid for the analysis of the consequences of large-scale advection of the turbulent microstructure. Since Eulerian frequencies larger than $(\epsilon/\nu)^{\frac{1}{2}}$ can be generated only by advective spectral broadening, and since the r.m.s. value of $\partial u/\partial t$ is determined by the position of the viscous cut-off in the Eulerian frequency spectrum, it appears reasonable to postulate that the Eulerian time microscale is determined by large-scale advection of dissipative eddies. This hypothesis serves as the starting point for further analysis.

2. Analysis

If Taylor's hypothesis governs the advection of dissipative eddies past a fixed observation point, we can write

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z}. \tag{1}$$

If we assume that the microstructure is statistically independent of the energy-containing eddies, we obtain for the mean-square value of (1) in isotropic turbulence without mean flow

$$\overline{\left(\frac{\partial u}{\partial t}\right)^2} = \overline{u^2}\,\overline{\left(\frac{\partial u}{\partial x}\right)^2} + \overline{v^2}\,\overline{\left(\frac{\partial u}{\partial y}\right)^2} + \overline{w^2}\,\overline{\left(\frac{\partial u}{\partial z}\right)^2}. \tag{2}$$

In isotropic turbulence, we have (Taylor 1935; Batchelor 1953, p. 110)

$$\overline{u^2} = \overline{v^2} = \overline{w^2}, \tag{3}$$

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = \frac{1}{2}\overline{\left(\frac{\partial u}{\partial y}\right)^2} = \frac{1}{2}\overline{\left(\frac{\partial u}{\partial z}\right)^2}. \tag{4}$$

Therefore, (2) reduces to

$$\overline{\left(\frac{\partial u}{\partial t}\right)^2} = 5\overline{u^2}\,\overline{\left(\frac{\partial u}{\partial x}\right)^2}. \tag{5}$$

In isotropic turbulence, the following relation holds (Taylor 1935; Tennekes & Lumley 1972, p. 66):

$$\epsilon = 15\nu\overline{\left(\frac{\partial u}{\partial x}\right)^2}. \tag{6}$$

Also, the Taylor microscale λ is defined by (Taylor 1935)

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = \frac{\overline{u^2}}{\lambda^2}. \tag{7}$$

Therefore, (5) may be written as

$$\overline{\left(\frac{\partial u}{\partial t}\right)^2} = \frac{1}{3}\overline{u^2}\frac{\epsilon}{\nu}, \tag{8}$$

or as

$$\overline{\left(\frac{\partial u}{\partial t}\right)^2} = 5\frac{(\overline{u^2})^2}{\lambda^2}. \tag{9}$$

Comte-Bellot & Corrsin (1971) define the Eulerian time microscale τ_E through the relation†

$$\overline{\left(\frac{\partial u}{\partial t}\right)^2} = \frac{2\overline{u^2}}{\tau_E^2}. \tag{10}$$

Substitution of (8) and (9) into (10) yields

$$\tau_E = \left(\frac{2}{5}\frac{\lambda^2}{\overline{u^2}}\right)^{\frac{1}{2}} = \left(\frac{6\nu}{\epsilon}\right)^{\frac{1}{2}}. \tag{11}$$

In the experiments by Comte-Bellot & Corrsin, the value of τ_E was determined from the Eulerian time correlation behind a two-inch grid, with the origin of the time delay chosen at the point $x/M = 42$ (M is the mesh size of the grid). Their experimental value for τ_E was 6·2 ms. At the reference position, λ was equal to 0·484 cm and $(\overline{u^2})^{\frac{1}{2}}$ was 22·2 cm s^{-1}. Substituting these values into (11), we find that the predicted value of τ_E is 13·8 ms.

If τ_E is estimated on basis of the hypothetical similarity between the Eulerian and Lagrangian frequency spectra (Corrsin 1963), a value of approximately 30 ms is obtained (Comte-Bellot & Corrsin 1971). The Lagrangian time microscale in the same flow was measured by Shlien & Corrsin (1974); its value is 76 ms. It appears that calculations based on the advection hypothesis are more realistic than calculations based on the Eulerian–Lagrangian similarity hypothesis.

The measured value of τ_E is about half as big as our present estimate. This discrepancy deserves comment. Close inspection of the data collected by Comte-Bellot & Corrsin reveals that the shape of the Eulerian correlation curve near the origin is based on questionable extrapolations. The smallest time delay used was 1·5 ms; there is no experimental evidence that the correlation curve is parabolic near the origin. Also, the correlation coefficients for small time delays were obtained by extrapolation from measurements taken with hot wires that were slightly displaced sideways. It seems likely that this procedure underestimates the correlation coefficients at small time delays. This would result in an underestimate of τ_E. Since the correlation coefficients involved are so close to one, the

† There is a misprint in the formula used by Comte-Bellot & Corrsin.

true experimental value of τ_E is probably substantially larger than the estimate given by Comte-Bellot & Corrsin.

It should also be pointed out that the low Reynolds number grid turbulence used in the Comte-Bellot & Corrsin experiments is not an ideal test case for the advection hypothesis. At low Reynolds numbers, the advective spectral broadening is not very pronounced, and the validity of Taylor's hypothesis is questionable. Still, the relatively good agreement between the prediction based on the advection hypothesis and the experimental result is sufficiently encouraging to attempt an alternative analysis of the Eulerian frequency spectrum in 'box' turbulence.

3. The Eulerian frequency spectrum

The frequency spectrum observed at a fixed point in isotropic turbulence without a mean flow is strongly affected by advective spectral broadening. At a frequency corresponding to the viscous cut-off in the Lagrangian time spectrum for example, fluctuations are observed which are related to the passage of eddies in the inertial subrange. Some qualitative estimates will help to illustrate the issue. Large-scale advection of eddies of size r (where r is taken to be in the inertial subrange) creates frequencies of order q/r. In order to find the value of r that contributes most to the energy at the Eulerian frequency corresponding to the cut-off frequency $(\epsilon/\nu)^{\frac{1}{2}}$ of the Lagrangian spectrum, we have to put

$$q/r \sim (\epsilon/\nu)^{\frac{1}{2}}. \tag{12}$$

This yields

$$r \sim q(\nu/\epsilon)^{\frac{1}{2}}. \tag{13}$$

If r relates to an eddy in the inertial subrange, its kinetic energy may be estimated as (Tennekes & Lumley 1972, p. 260)

$$\tfrac{1}{2}u^2(r) \sim \epsilon^{\frac{2}{3}}r^{\frac{2}{3}}. \tag{14}$$

The contribution of these eddies to the kinetic energy at the Eulerian frequency $(\epsilon/\nu)^{\frac{1}{2}}$ is therefore

$$\tfrac{1}{2}u^2\{r, (\epsilon/\nu)^{\frac{1}{2}}\} \sim \epsilon^{\frac{1}{3}}q^{\frac{2}{3}}\nu^{\frac{1}{3}}. \tag{15}$$

In the absence of advection by large scales, the kinetic energy at a frequency $(\epsilon/\nu)^{\frac{1}{2}}$ would be

$$\tfrac{1}{2}u^2\{\eta, (\epsilon/\nu)^{\frac{1}{2}}\} \sim (\nu\epsilon)^{\frac{1}{2}}. \tag{16}$$

The ratio of (15) and (16) is

$$\frac{\tfrac{1}{2}u^2\{r, (\epsilon/\nu)^{\frac{1}{2}}\}}{\tfrac{1}{2}u^2\{\eta, (\epsilon/\nu)^{\frac{1}{2}}\}} \sim \left(\frac{q^2}{(\epsilon\nu)^{\frac{1}{2}}}\right)^{\frac{1}{3}}. \tag{17}$$

Here, $(\epsilon\nu)^{\frac{1}{2}}$ is the kinetic energy of the dissipative eddies. Clearly, the advective contribution outweighs the quasi-Lagrangian one, at least if the Reynolds number of the turbulence is large enough.

We conclude that the high frequency end of the Eulerian time spectrum must be dominated by the Doppler shifts in frequency caused by random advection by the energy-containing eddies. This generalization of the advection hypothesis permits us to obtain a probable form for the inertial subrange (obviously better referred to as the inertial–advective subrange) in the Eulerian frequency spectrum.

If the dominant contribution to the kinetic energy at a frequency ω in the inertial–advective subrange is made by large-scale advection of eddies in the inertial subrange of the wavenumber spectrum, we have

$$\omega \sim q/r \tag{18}$$

and

$$\tfrac{1}{2}u^2(\omega) \sim \epsilon^{\frac{2}{3}} r^{\frac{2}{3}}. \tag{19}$$

Substitution of (18) into (19) yields

$$\tfrac{1}{2}u^2(\omega) \sim \epsilon^{\frac{2}{3}} q^{\frac{2}{3}} \omega^{-\frac{2}{3}}. \tag{20}$$

The Eulerian frequency spectrum is defined as the kinetic energy per unit frequency. We obtain

$$\Phi_E(\omega) = \beta_E \epsilon^{\frac{2}{3}} q^{\frac{2}{3}} \omega^{-\frac{5}{3}}, \tag{21}$$

where β_E is an unknown constant, which presumably is of order one.

The inertial–advective subrange in the Eulerian frequency spectrum thus does not obey Kolmogorov scaling, and is markedly different from the inertial subrange in the Lagrangian frequency spectrum. The latter is (Inoue 1951; Corrsin 1963; Tennekes & Lumley 1972, p. 277)

$$\Phi_L(\omega) = \beta_L \epsilon \omega^{-2}. \tag{22}$$

Let us compare (21) and (22) at the lowest frequency for which they might give a reasonable representation. That frequency is $\omega \sim q/l$ (l is an integral scale), and we find

$$\frac{\Phi_E(q/l)}{\Phi_L(q/l)} = \frac{\beta_E}{\beta_L}\left(\frac{q^3/l}{\epsilon}\right)^{\frac{1}{3}}. \tag{23}$$

Since $\epsilon \sim q^3/l$, the values of Φ_E and Φ_L at the large-scale end are of comparable magnitude. In the absence of high Reynolds number data on these spectra, we cannot determine whether the Eulerian spectrum is likely to have a ω^{-2} shape at frequencies below those in the inertial–advective subrange, but it seems fair to speculate that such a small difference in spectral slope would be extremely hard to verify experimentally. One point appears to be clear, however: since the spectral 'smearing' caused by random advection tends to remove discontinuities in the spectral slope, the inertial–advective spectrum proposed here may well be a valid approximation at frequencies near those characteristic of the large-scale structure.

The spectra given by (21) and (22) can, after multiplication by ω^2, be integrated to obtain estimates for the mean-square values of $\partial u/\partial t$ and du/dt:

$$\overline{\left(\frac{\partial u}{\partial t}\right)^2} = \int_0^\infty \omega^2 \Phi_E(\omega)\, d\omega = \int_0^\infty \beta_E \epsilon^{\frac{2}{3}} q^{\frac{2}{3}} \omega^{\frac{1}{3}}\, d\omega \cong \tfrac{3}{4}\beta_E \epsilon^{\frac{2}{3}} q^{\frac{2}{3}} \omega_{E,D}^{\frac{4}{3}} \tag{24}$$

and

$$\overline{\left(\frac{du}{dt}\right)^2} = \int_0^\infty \omega^2 \Phi_L(\omega)\, d\omega = \int_0^\infty \beta_E \epsilon\, d\omega \cong \beta_E \epsilon \omega_{L,D}. \tag{25}$$

Here, $\omega_{E,D}$ is the frequency of the viscous cut-off in the Eulerian spectrum and $\omega_{L,D}$ is its Lagrangian counterpart. According to the advection hypothesis,

$$\omega_{E,D} = q/\eta = q\epsilon^{\frac{1}{4}}\nu^{-\frac{3}{4}}, \tag{26}$$

while the highest Lagrangian frequency is

$$\omega_{L,D} = (\epsilon/\nu)^{\frac{1}{2}}. \tag{27}$$

In (26) and (27), unknown numerical coefficients have been ignored. Substitution of (26) into (24) yields

$$\overline{\left(\frac{\partial u}{\partial t}\right)^2} = C_E^2 q^2 \left(\frac{\epsilon}{\nu}\right). \tag{28}$$

The value of C_E^2 can be estimated by comparing (28) with (8); this yields $C_E = \frac{1}{3}$, because $q^2 = 3\overline{u^2}$.

Substitution of (27) into (25) yields

$$\overline{\left(\frac{du}{dt}\right)^2} = C_L^2 \left(\frac{\epsilon^3}{\nu}\right)^{\frac{1}{2}}. \tag{29}$$

The value of C_L can be estimated from the data given by Shlien & Corrsin (1974). They define the Lagrangian time microscale by†

$$\overline{\left(\frac{du}{dt}\right)^2} = \frac{2\overline{u^2}}{\tau_L^2}. \tag{30}$$

Substitution of (30) into (29) gives

$$\tau_L = \frac{2^{\frac{1}{2}}(\overline{u^2})^{\frac{1}{2}}}{C_L(\epsilon^3/\nu)^{\frac{1}{4}}}. \tag{31}$$

For $(\overline{u^2})^{\frac{1}{2}} = 22{\cdot}2\,\mathrm{cm\,s^{-1}}$, $\epsilon = 0{\cdot}4740\,\mathrm{m^2\,s^{-3}}$ and $\nu = 15\times 10^{-6}\,\mathrm{m^2\,s^{-1}}$, Shlien & Corrsin found $\tau_L = 76\times 10^{-3}\,\mathrm{s}$. Substitution of these values into (31) gives $C_L = \frac{4}{9}$ approximately.

The ratio between the Eulerian time microscale and its Lagrangian counterpart thus is given by

$$\frac{\tau_E}{\tau_L} = \frac{4}{3}\frac{(\epsilon\nu)^{\frac{1}{4}}}{q}. \tag{32}$$

Here we have used $C_E = \frac{1}{3}$ and $C_L = \frac{4}{9}$.

The velocity $(\epsilon\nu)^{\frac{1}{4}}$ occurring in (32) is the Kolmogorov velocity of the dissipative eddies. Since the ratio $(\epsilon\nu)^{\frac{1}{4}}/q$ is proportional to $R_l^{-\frac{1}{4}} = (ql/\nu)^{-\frac{1}{4}}$, this result confirms that the Eulerian time microscale must be appreciably smaller than its Lagrangian counterpart if the Reynolds number of the turbulence is large enough. The comparison also shows that the approximate equality of τ_E and τ_L predicted by Corrsin's (1963) Eulerian–Lagrangian similarity hypothesis is bound to produce unrealistic values of τ_E. The values of τ_E and τ_L obtained in the experiments by Corrsin and his co-workers prove that the advection hypothesis is justified, even at relatively low Reynolds numbers.

† There is a misprint in the formula used by Shlien & Corrsin.

4. Discussion

The consequences of the advection hypothesis are rather embarrassing in a personal sense. The section on time spectra in chapter 8 of Tennekes & Lumley (1972) treats the Eulerian spectrum on basis of the similarity hypothesis; if the analysis presented in this paper proves to be reliable, that section will have to be revised before a new edition goes to press.

The advection-dominated Eulerian spectrum strongly suggests that the evolution of turbulence in wavenumber space is best computed on a Lagrangian basis. Large-scale advection of the small-scale structure creates Eulerian Fourier components at frequencies that are higher than the angular velocities characterizing the internal evolution of the scales being advected, and calculations of the temporal evolution at the points of an Eulerian grid would tend to get overwhelmed by these spurious advection effects. From this point of view, models such as Kraichnan's Lagrangian-history direct-interaction approximation obviously are to be preferred over their Eulerian counterparts.

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REFERENCES

BATCHELOR, G. K. 1953 *The Theory of Homogeneous Turbulence.* Cambridge University Press.

COMTE-BELLOT, G. & CORRSIN, S. 1971 Simple Eulerian time correlation of full- and narrow-band velocity signals in grid-generated, isotropic turbulence. *J. Fluid Mech.* **48**, 273–337.

CORRSIN, S. 1963 Estimates of the relations between Eulerian and Lagrangian scales in large Reynolds number turbulence. *J. Atmos. Sci.* **20**, 115–119.

FAVRE, A., GAVIGLIO, J. & DUMAS, R. 1951 Mesures de la correlation dans le temps et l'espace et spectres de la turbulence en soufflerie. *Publ. Sci. Tech. Ministère Air,* **251**, 293–309.

INOUE, E. 1951 On turbulent diffusion in the atmosphere. *J. Met. Soc. Japan,* **29**, 246–252.

SHLIEN, D. J. & CORRSIN, S. 1974 A measurement of Lagrangian velocity autocorrelation in approximately isotropic turbulence. *J. Fluid Mech.* **62**, 255–271.

TAYLOR, G. I. 1935 Statistical theory of turbulence. *Proc. Roy. Soc.* A **151**, 421–478.

TENNEKES, H. & LUMLEY, J. L. 1972 *A First Course in Turbulence.* M.I.T. Press.